ON THE CAUCHY PROBLEM FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE OVER THE FIELD OF p-ADIC NUMBERS. II. [1]

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1. Let $\mathfrak{B}$ be a Banach space with norm $\|\cdot\|$ over the completion $\Omega = \Omega_p$ of an algebraic closure of the field Q_p of p-adic numbers (p is prime) (for details we refer to [1 - 3]), and let A be a closed linear operator on $\mathfrak{B}$.

For a number $\alpha > 0$, we put

$$E_\alpha(A) = \Big\{x \in \bigcap_{n\in\mathbb{N}_0=\{0,1,2,\dots\}} \mathcal{D}(A^n) \Big| \exists c = c(x) > 0 \quad \forall k \in \mathbb{N}_0 \quad \|A^k x\| \le c\alpha^k\Big\}$$

($\mathcal{D}(A)$ is the domain of A).

The linear set $E_\alpha(A)$ is a Banach space with respect to the norm

$$\|x\|_\alpha = \sup_{n\in\mathbb{N}_0} \frac{\|A^n x\|}{\alpha^n}.$$

Denote by $E(A)$ the space of entire vectors of exponential type for the operator A:

$$E(A) = \operatorname*{ind\,lim}_{\alpha\to\infty} E_\alpha(A).$$

So, as a set,

$$E(A) = \bigcup_{\alpha>0} E_\alpha(A).$$

By the type $\sigma(x; A)$ of a vector $x \in E(A)$ we mean the number

$$\sigma(x; A) = \inf\{\alpha > 0 : x \in E_\alpha(A)\} = \overline{\lim_{n\to\infty}} \|A^n x\|^{\frac{1}{n}}. \tag{1}$$

In the case, where $\mathcal{D}(A) = \mathfrak{B}$, i.e., the operator A is bounded, $E(A) = \mathfrak{B}$, and

$$\forall x \in \mathfrak{B} \quad \sigma(x; A) \le \|A\|.$$

2. In what follows we shall deal with power series of the form

$$y(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathfrak{B}, \quad z \in \Omega. \tag{2}$$

For such a series the convergence radius is determined by the formula

$$r = r(y) = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\|c_n\|}}. \tag{3}$$

If $r(y) > 0$, then series (2) gives a vector-valued function $y(z)$ with values in $\mathfrak{B}$ (a $\mathfrak{B}$-valued function) in the open disk $U_r^-(0) = \{z \in \Omega : |z|_p < r\}$ ($|\cdot|_p$ is the p-adic valuation on Ω).

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For a number $r > 0$, we denote by $\mathfrak{A}_r(\mathfrak{B})$ the set of all $\mathfrak{B}$-valued functions $y(z)$ which satisfy the following conditions:

(i) $y(z)$ is of the form (2) with $r(y) \geq r$;

(ii) $\lim\limits_{i\to\infty} \|c_i\| r^i = 0$.

The linear set $\mathfrak{A}_r(\mathfrak{B})$ is a Banach space with respect to the norm

$$\|y\|_r = \sup_{n\in\mathbb{N}_0} \|c_n\| r^n.$$

Moreover, if $0 < r_1 < r$, then the embedding $\mathfrak{A}_r(\mathfrak{B}) \hookrightarrow \mathfrak{A}_{r_1}(\mathfrak{B})$ induced by restriction of the domain of a vector-valued function is continuous.

We put

$$\mathfrak{A}_{loc}(\mathfrak{B}) = \operatorname{ind}\lim_{r\to 0} \mathfrak{A}_r(\mathfrak{B}).$$

The space $\mathfrak{A}_{loc}(\mathfrak{B})$ is called the space of locally analytic at zero $\mathfrak{B}$-valued functions.

It follows from (3) that for the convergence radius of i-order derivative

$$y^{(i)}(z) = \sum_{n=0}^{\infty} (n+1)\ldots(n+i) c_{n+i} z^n, \quad i \in \mathbb{N},$$

of a vector-valued function $y(z)$ from $\mathfrak{A}_{loc}(\mathfrak{B})$, valid is the inequality

$$r(y^{(i)}) \geq r(y).$$

It is not also hard to check that if $z \to 0$ in Ω, then

$$y(z) \to y(0) = c_0, \quad \frac{y^{(i)}(z) - y^{(i)}(0)}{z} \to y^{(i+1)}(0) = c_{i+1}(i+1)!$$

in the topology of the space $\mathfrak{B}$.

3. Let $x \in E(A)$, $\sigma(x; A) = \sigma$. For a fixed natural m we consider the Mittag-Leffler $\mathfrak{B}$-valued functions

$$F_k(z; A)x = \sum_{n=0}^{\infty} \frac{z^{mn+k} A^n x}{(mn+k)!}, \quad z \in \Omega, \quad k = 0, 1, \ldots, m-1. \tag{4}$$

Proposition 1. *The convergence radius of series* (4) *does not depend on k, and it is determined by the formula*

$$r(F_k(\cdot; A)x) = r = \sigma^{-\frac{1}{m}} p^{-\frac{1}{p-1}}. \tag{5}$$

Proof. According to (3),

$$r^{-1}(F_k(\cdot; A)x) = \overline{\lim_{n\to\infty}} \sqrt[mn+k]{\frac{\|A^n x\|}{|(mn+k)!|_p}}.$$

By (1), for any $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}_0$,

$$\|A^n x\| \leq (\sigma + \varepsilon)^n, \tag{6}$$

and there exists a subsequence $n_i \to \infty$ ($i \to \infty$) such that

$$\lim_{i\to\infty} \frac{\|A^{n_i}x\|}{(\sigma-\varepsilon)^{n_i}} = \infty. \tag{7}$$

Using (6) and the estimate

$$\frac{1}{np} p^{\frac{n}{p-1}} \le \frac{1}{|n!|_p} \le p^{\frac{n-1}{p-1}} \tag{8}$$

valid for large $n \in \mathbb{N}$ (see [4]), we obtain

$$\overline{\lim_{n\to\infty}} \sqrt[mn+k]{\frac{\|A^n x\|}{|(mn+k)!|_p}} \le (\sigma+\varepsilon)^{\frac{n}{mn+k}} p^{\frac{mn+k-1}{(mn+k)(p-1)}}.$$

Since

$$\lim_{n\to\infty} (\sigma+\varepsilon)^{\frac{n}{mn+k}} p^{\frac{mn+k-1}{(mn+k)(p-1)}} = (\sigma+\varepsilon)^{\frac{1}{m}} p^{\frac{1}{(p-1)}},$$

the inequality

$$\overline{\lim_{n\to\infty}} \sqrt[mn+k]{\frac{\|A^n x\|}{|(mn+k)!|_p}} \le (\sigma+\varepsilon)^{\frac{1}{m}} p^{\frac{1}{(p-1)}}$$

holds. Taking into account that $\varepsilon > 0$ is arbitrary, we conclude that

$$\overline{\lim_{n\to\infty}} \sqrt[mn+k]{\frac{\|A^n x\|}{|(mn+k)!|_p}} \le \sigma^{\frac{1}{m}} p^{\frac{1}{(p-1)}},$$

whence

$$r(F_k(\cdot;A)x) \ge \sigma^{-\frac{1}{m}} p^{-\frac{1}{(p-1)}}.$$

If now $\{n_i\}_{i=1}^{\infty}$ is a subsequence satisfying (7), then because of (8),

$$\sqrt[mn_i+k]{\frac{\|A^{n_i}x\|}{|(mn+k)!|_p}} \ge (\sigma-\varepsilon)^{\frac{n_i}{mn_i+k}} p^{\frac{1}{(p-1)}} [p(mn_i+k)]^{-\frac{1}{(mn_i+k)}}.$$

Since

$$\lim_{n\to\infty} (\sigma-\varepsilon)^{\frac{n_i}{mn_i+k}} [p(mn_i+k)]^{-\frac{1}{(mn_i+k)}} = (\sigma-\varepsilon)^{\frac{1}{m}},$$

we have

$$\overline{\lim_{n\to\infty}} \sqrt[mn+k]{\frac{\|A^n x\|}{|(mn+k)!|_p}} \ge (\sigma-\varepsilon)^{\frac{1}{m}} p^{\frac{1}{(p-1)}},$$

whence

$$r(F_k(\cdot;A)x) \le \sigma^{-\frac{1}{m}} p^{-\frac{1}{(p-1)}}.$$

Thus, formula (5) is valid. Q.E.D..

One can easily check that for the derivatives of $F_k(z;A)x)$, the relations

$$F_k^{(i)}(z;A)x) = \begin{cases} F_{k-i}(z;A)x & \text{if } \ 0 \le i \le k \\ F_0^{(i-k)}(z;A)x & \text{if } \ i > k \end{cases} \tag{9}$$

and

$$\begin{cases} F_0^{(ml)}(z;A)x &=& F_0(z;A)A^l x \\ F_0^{(ml+j)}(z;A)x &=& (F_0(z;A)A^l x)^{(j)} = F_{m-j}(z;A)A^{l+1}x \ 1 \le j \le m-1 \end{cases} \tag{10}$$

are fulfilled, which imply, with regard to $\sigma(A^j x; A) = \sigma(x; A)$ $(j \in \mathbb{N})$, that

$$r(F_k^{(i)}(\cdot; A)x) = \sigma^{-\frac{1}{m}} p^{-\frac{1}{(p-1)}} \quad i \in \mathbb{N}_0,\ k = 0, 1, \ldots, m-1.$$

4. Let us consider the Cauchy problem

$$\begin{cases} y^{(m)}(z) &= Ay(z) \\ y^{(k)}(0) &= y_k, \quad k = 0, 1, \ldots, m-1, \end{cases} \tag{11}$$

where A is a closed linear operator on $\mathfrak{B}$ (the case $m = 1$ was discussed in [5]). By a solution of this problem we mean a vector-valued function of the form (2) with values in $\mathcal{D}(A)$ that satisfies (11). The question arises, under what conditions on the initial data y_k problem (11) has a solution in the space $\mathfrak{A}_{loc}(\mathfrak{B})$. The following assertion gives an answer.

Theorem 1. *Problem* (11) *is solvable in the class* $\mathfrak{A}_{loc}(\mathfrak{B})$ *if and only if* $y_k \in E(A)$, $k = 0, 1, \ldots, m-1$. *The solution is represented in the form*

$$y(z) = \sum_{k=0}^{m-1} F_k(z, A) y_k. \tag{12}$$

Moreover, problem (11) *is well-posed in* $\mathfrak{A}_{loc}(\mathfrak{B})$*, that is, the solution is unique in this space, and the convergence* $E(A) \ni y_{i,k} \to y_k$ $(i \to \infty,\ k = 0, 1, \ldots, m-1)$ *in the* $E(A)$*-topology implies the convergence of the sequence of corresponding solutions* $y_i(z)$ *to* $y(z)$ *in* $\mathfrak{A}_{loc}(\mathfrak{B})$.

Proof. Assume that problem (11) is solvable in $\mathfrak{A}_{loc}(\mathfrak{B})$. This means that there exists a $\mathcal{D}(A)$-valued function $y(z) = \sum\limits_{n=0}^{\infty} c_n z^n \in \mathfrak{A}_r(\mathfrak{B})$ with some $r > 0$ $(c_n \in \mathfrak{B},\ z \in \Omega)$, which satisfies (11). It is obvious that $y_0 = y(0) \in \mathcal{D}(A)$. Further, if we take $|\Delta z|_p < |z|_p$, then $|z + \Delta z|_p = |z|_p$, the inequality $|z|_p < r(y)$ implies $|z + \Delta z|_p < r(y)$, and for any $z \in U_r^-(0)$ we have

$$\frac{y^{(m)}(z + \Delta z) - y^{(m)}(z)}{\Delta z} = A \frac{y(z + \Delta z) - y(z)}{\Delta z}.$$

In view of closedness of the operator A and the fact [3] that

$$\frac{y^{(k)}(z + \Delta z) - y^{(k)}(z)}{\Delta z} \to y^{(k+1)}(z),$$

the values of $y'(z)$ in $U_r^-(0)$ belong to $\mathcal{D}(A)$, and

$$y^{(m+1)}(z) = Ay'(z) \quad \text{when } |z|_p < r(y).$$

Repeating this procedure as much as we need, we arrive at the inclusion that

$$y^{(n)}(z) \in \mathcal{D}(A) \ \text{ if } \ |z|_p < r(y) \le r, \quad n \in \mathbb{N}_0,$$

and at the equality

$$y^{(mn+k)}(z) = A^n y^{(k)}(z), \quad |z|_p < r(y) \le r, \quad k = 0, 1, \ldots, m-1,$$

whence

$$A^n y_k = y^{(mn+k)}(0) = (mn+k)! c_{mn+k} \tag{13}$$

By property (ii) of a $\mathfrak{B}$-valued function $y \in \mathfrak{A}_r(\mathfrak{B})$ and the estimate (8),

$$\|A^n y_k\| = |(mn+k)!|_p \|c_{mn+k}\| \leq \frac{(mn+k)p\|c_{mn+k}\|p^{-\frac{mn+k}{p-1}}}{r^{mn+k}} r^{mn+k} \leq c\alpha^n,$$

where

$$c = p\left(\frac{(1+\varepsilon)p^{-\frac{1}{p-1}}}{r}\right)^k \sup_i \|c_i\| r^i, \quad \alpha = \left(\frac{(1+\varepsilon)p^{-\frac{1}{p-1}}}{r}\right)^m$$

($\varepsilon > 0$ is arbitrary). So, $y_k \in E(A),\ k = 0, 1, \ldots, m-1$.

Conversely, let $y_k \in E(A),\ k = 0, 1, \ldots, m-1$. Taking into account relations (9) and (10), one can verify that the $\mathfrak{B}$-valued functions $F_k(z; A)x,\ x \in E(A)$, satisfy the equation from (11) in $U_r^-(0)$ with $r = \sigma(x; A)^{-\frac{1}{m}} p^{-\frac{1}{p-1}}$ and the initial data

$$F_k^{(i)}(0; A)x = \delta_{ik} x, \quad i, k = 0, 1, \ldots, m-1.$$

Therefore the $\mathcal{D}(A)$-valued function (12) is a solution of (11) from the space $\mathfrak{A}_{loc}(\mathfrak{B})$. It follows from (13) that this solution is unique in the mentioned class.

To prove the well-posedness of problem (11) in $\mathfrak{A}_{loc}(\mathfrak{B})$, assume that a sequence $y_{i,k} \in E(A)$ converges in $E(A)$ to y_k as $i \to \infty,\ k = 0, 1, \ldots, m-1$. This means that there exists $\alpha > 0$ such that $y_{i,k}, y_k \in E_\alpha(A)$, and

$$\|y_{i,k} - y_k\|_\alpha \to 0 \ \text{ as } i \to \infty.$$

Then the corresponding solutions $y_i(z)$ and $y(z)$ of problem (11) belong to $\mathfrak{A}_r(\mathfrak{B})$ where $r < \alpha^{-\frac{1}{m}} p^{-\frac{1}{p-1}}$, and

$$\|y_i(\cdot) - y(\cdot)\|_r \leq \sum_{k=0}^{m-1} \|F_k(\cdot; A)y_k\|_r = \sum_{k=0}^{m-1} \sup_{n\in\mathbb{N}_0} \left\|\frac{A^n(y_k - y_{i,k})}{(mn+k)!}\right\| r^{mn+k} \leq$$

$$\sum_{k=0}^{m-1} \sup_{n\in\mathbb{N}_0} \frac{\alpha^n \|y_k - y_{i,k}\|_\alpha \alpha^{-\frac{mn+k}{m}} p^{-\frac{mn+k}{p-1}}}{|(mn+k)!|_p} \leq$$

$$\sum_{k=0}^{m-1} \sup_{n\in\mathbb{N}_0} \alpha^{-\frac{k}{m}} p^{-\frac{mn+k}{p-1}} p^{\frac{mn+k-1}{p-1}} \|y_{i,k} - y_k\|_\alpha =$$

$$\sum_{k=0}^{m-1} \alpha^{-\frac{k}{m}} p^{-\frac{1}{p-1}} \|y_{i,k} - y_k\|_\alpha = p^{-\frac{1}{p-1}} \sum_{k=0}^{m-1} \alpha^{-\frac{k}{m}} \|y_{i,k} - y_k\|_\alpha.$$

Since $\|y_{i,k} - y_k\|_\alpha \to 0$ as $i \to \infty$ $(k = 0, 1, \ldots, m-1)$, we have

$$\|y_i(\cdot) - y(\cdot)\|_r \to 0 \ \ (i \to \infty),$$

which completes the proof.

It follows from the above proof, that in order that problem (11) be solvable in the class of entire $\mathfrak{B}$-valued functions, it is necessary and sufficient that

$$y_k \in \bigcap_{\alpha>0} E_\alpha(A).$$

Corollary 1. *If the operator A is bounded, then problem* (11) *is well-posed in $\mathfrak{A}_r(\mathfrak{B})$ for any $y_k \in \mathfrak{B}$.*

Remark 1. *As is shown in* [6]*, in the case where $\mathfrak{B}$ is a Banach space over the field $\mathbb{C}$ of complex numbers, the Cauchy problem* (11) *is well-posed in the class $\mathfrak{A}_r(\mathfrak{B})$ if and only if*

$$\forall \alpha > 0 \quad \exists c = c(\alpha) > 0 \quad \|A^n y_k\| \leq c\alpha^n n^{mn}, \quad k = 0, 1, \ldots, m-1.$$

In order that the solution of (11) *be an entire $\mathfrak{B}$-valued function of exponential type, it is necessary and sufficient that $y_k \in E(A),\ 0, 1, \ldots, m-1$.*

5. In this section we show how the result of Theorem 1 can be applied to partial differential equations (see also [7]).

Let $\mathcal{A}_\rho$ be the space of Ω-valued functions $f(x)$ analytic on the n-dimensional disk

$$U_\rho^+(0) = \left\{ x = (x_1, \ldots x_n) \in \Omega^n : |x|_p = \left(\sum_{i=1}^n |x_i|_p^2 \right)^{1/2} \leq \rho \right\}.$$

This means that

$$f(x) = \sum_\alpha f_\alpha x^\alpha, \quad f_\alpha \in \Omega, \quad \lim_{|\alpha| \to \infty} |f_\alpha|_p \rho^{|\alpha|} = 0,$$

where $\alpha = (\alpha_1, \ldots \alpha_n),\ \alpha_i \in \mathbb{N}_0,\ |\alpha| = \alpha_1 + \cdots + \alpha_n$.

The space $\mathcal{A}_\rho$ is a p-adic Banach space with respect to the norm

$$\|f\|_\rho = \sup_\alpha |f_\alpha|_p \rho^{|\alpha|}.$$

It is clear that the differential operators

$$f \mapsto \frac{\partial f}{\partial x_j} = \sum_\alpha \alpha_j f_\alpha x_1^{\alpha_1} \ldots x_j^{\alpha_j - 1} \ldots x_n^{\alpha_n},\ j = 1, \ldots, n,$$

are bounded in $\mathcal{A}_\rho$, and

$$\left\| \frac{\partial f}{\partial x_j} \right\|_\rho = \sup_\alpha |f_\alpha \alpha_j|_p \rho^{|\alpha|-1} \leq \frac{1}{\rho} \sup_\alpha |f_\alpha|_p \rho^{|\alpha|} = \frac{1}{\rho} \|f\|_\rho.$$

Since for $f(x) = x_j$, $\left\| \frac{\partial f}{\partial x_j} \right\|_\rho = \frac{1}{\rho}$, the norm of the operator $\frac{\partial}{\partial x_j}$ is equal to $\frac{1}{\rho}$.

The multiplication operator

$$G : f \mapsto fg,\ f \in \mathcal{A}_\rho, \quad g \in \mathcal{A}_\rho,$$

is bounded in $\mathcal{A}_\rho$, too, and

$$\|G\| = \|g\|_\rho.$$

Indeed, let $f(x) = \sum_\alpha f_\alpha x^\alpha,\ g(x) = \sum_\alpha g_\alpha x^\alpha,\ \alpha = (\alpha_1, \ldots, \alpha_n)$. Then

$$f(x)g(x) = \sum_\alpha c_\alpha x^\alpha,$$

where

$$c_\alpha = \sum_{0\le i\le\alpha} f_i g_{\alpha-i} = \sum_{i_1=0}^{\alpha_1}\cdots\sum_{i_n=0}^{\alpha_n} f_{i_1,\dots,i_n} g_{\alpha_1-i_1,\dots,\alpha_n-i_n} \quad (i=(i_1,\dots,i_n)).$$

So,

$$\|fg\|_\rho = \sup_\alpha \max_{0\le i\le\alpha} |f_i|_p |g_{\alpha-i}|_p \rho^{|i|}\rho^{|\alpha|-|i|} \le \|f\|_\rho \|g\|_\rho.$$

As $\|fg\|_\rho = \|g\|_\rho$ for $f\equiv 1$, we have $\|G\| = \|g\|_\rho$.

Let us consider now the Cauchy problem

$$\begin{cases} \dfrac{\partial^m u(t,x)}{\partial t^m} &= \sum\limits_{|\beta|=0}^{n} a_\beta(x) D^\beta u(t,x) \\ u^{(k)}(0,x) &= \varphi_k(x), \quad k=0,1,\dots,m-1, \end{cases} \tag{14}$$

where

$$a_\beta(x)\in\mathcal{A}_\rho, \quad \varphi_k(x)\in\mathcal{A}_\rho, \quad D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}.$$

If we put $\mathfrak{B} = \mathcal{A}_\rho$ and define the operator A as

$$f\mapsto Af = \sum_{|\beta|=0}^{n} a_\beta D^\beta f,$$

then problem (14) can be written in the form (11). Furthermore, the relations

$$\Big\|\sum_{|\beta|=0}^{n} a_\beta D^\beta f\Big\|_\rho \le \max_\beta \|a_\beta D^\beta f\|_\rho \le \max_\beta \|a_\beta\|_\rho \|D^\beta f\|_\rho \le \max_\beta\{\rho^{-|\beta|}\|a_\beta\|_\rho\}\|f\|_\rho$$

show that the operator A is bounded in $\mathcal{A}_\rho$, and

$$\|A\| \le \max_\beta\{\rho^{-|\beta|}\|a_\beta\|_\rho\}.$$

It follows from Corollary 1 that problem (14) is well-posed in $\mathfrak{A}_{loc}(\mathcal{A}_\rho)$ in the disk $t\in\Omega : |t|_p < p^{-\frac{1}{p-1}}(\max\limits_\beta \rho^{-|\beta|}\|a_\beta\|_\rho)^{-\frac{1}{m}}$.

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